2. TOLOKONNIKOV L.A., Modification of different-modulus elasticity theory, Mekhan. Polimerov, 2, 1969.
3. WESOLOWSKI Z., Elastic material with different elastic constants in two regions of variability of deformation, Arch. Mech. Stosow., 21, 4, 1969.
4. JONES R.M., Relationships connecting stresses and strains in materials with different elastic moduli under tension and compression, AiAA Jnl, 15, 1, 1977.
5. LOMAKIN E.V. and RABOTNOV YU.N., Elasticity theory relationships for an isotropic different-modulus body. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 6, 1978.
6. AMBARTSUMYAN S.A., Different-modulus Elasticity Theory, Nauka, Moscow, 1982.
7. BELL J. Experimental Principles of the Mechanics of Deformable Solids, Pts. 1 and 2. Nauka, Moscow, 1984.
8. IVANDAYEV A.I. and GUBAIDULLIN A.A., Investigation of the non-stationary efflux of a boiling fluid in a thermodynamically equilibrium approximation, Teplofizika Vysokikh Temperatur, 16, 3, 1978.
9. MASLOV V.P. and MOSOLOV P.P., Elasticity Theory for a Different-Modulus Medium, Izd. MIEM, Moscow, 1985.
10. MASLOV V.P. and MOSOLOV P.P., General theory of solutions of the equations of motion of a different-modulus elastic medium. PMM, 49, 3, 1985.
11. MASLOV V.P., MOLOSOV P.P. and SOSNINA E.V., On types of discontinuities of solutions of longitudinal, free, one-dimensional motions in a different-modulus elastic medium, Problems of the Non-linear Mechanics of a Continuous Medium, Valgus, Tallin, 1985.
12. SEDOV L.I., Mechanics of a Continuous Medium, 2, Nauka, Moscow, 1984.
13. BLAND D., Non-linear Dynamic Elasticity Theory, Mir, Moscow, 1972.
14. MANDEL J. Plastic waves in an unbounded three-dimensional medium, Mekhanika, periodic collection of translated foreign papers, 5, 1963.
15. LANDAU L.D. and LIFSHITS E.M., Theoretical Physics, 6, Hydrodynamics, Nauka, Moscow, 1986.
16. KULIKOVSKII A.G. and REUTOV V.A., Motion of solitary and periodic waves with an amplitude close to the limiting one in a fluid layer of slowly varying depth, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. Gaza, 6, 1976.

Translated by M.D.F.

PMM U.S.S.R., Vol.53, No.2, pp.235-241, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

SINGULARITIES OF THE INTERACTION OF A VIBRATING STAMP WITH AN INHOMOGENEOUS HEAVY BASE*

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A method is developed for studying the fundamental characteristics of the wave process on the surface of an initally isotropic prestressed elastic half-space caused by an oscillating rigid stamp. The following is taken as the model of the inhomogeneous medium: an elastic layer $0 \leqslant x_3 \leqslant h$, $x_1, x_2 < \infty$ whose mechanical characteristics as well as the initial stresses are arbitrary, fairly smooth functions of the coordinate x_3 in the general case, lies on the surface of a homogeneous half-space $x_3 \geqslant h$, x_1, $x_2 < \infty$ (x_1, x_2, x_3 are a rectangular Cartesian coordinate system). The linearized boundary value problem of the dynamic theory of elasticity of vibrations with frequency ω for a rigid stamp on the surface of an inhomogeneous medium reduces to an integral equation or to

**Prikl.Matem.Mekhan.*, Vol.53, 2, 301-308, 1989

a system of integral equations of the first kind whose integral operator kernel is constructed numerically. Approximation of the kernel of the integral operator by a special kind of function enables an approximate solution of the integral equation to be constructed by the factorization method /1, 2/. On the basis of the latter, an effective investigation of the influence of the parameters characterising the inhomogeneity of the medium and the initial state of stress on the wave process both under (stress wave) and outside the stamp is possible.

The construction of a general linearized theory and the regularities of elastic wave propagation in bodies with homogeneous initial stresses are considered in /3/, where a fairly complete survey is also given of the literature on this question. A systematic exposition of the theory of wave propagation in elastic media with an inhomogeneous initial state was first given in /4/.

The contact problem of the vibrations of an inhomogeneous half-space subjected to a rigid stamp oscillating on its surface was examined in /5, 6/ without taking the initial stresses into account. A method of investigating the regularities of electric-wave excitation in semibounded bodies (layers and cylinders) with varying properties and magnitude of the initial stresses was proposed in /7/ on the basis of the solution of the contact problem. A method of investigating the singularities of elastic wave propagation in an inhomogeneous initially strained half-space caused by an oscillating load distributed in a certain domain on the surface of the medium has been developed*. (*Kalinchuk V.V., Lysenko I.V. and Polyakova I.B., Singularities of elastic wave excitation and propagation in an inhomogeneous heavy half-space. Deposited Manuscript, December 10, 1986, 8877-B86, VINITI, Rostov, 1986).

1. The problem of the vibrations of a rigid stamp occupying a domain Ω in planform on the surface of an elastic inhomogeneous initially strained medium is described by the linearized equations of motion /8, 9/

$$\nabla\cdot\boldsymbol{\theta}^{(i)}=\rho^{(i)}\mathbf{u}^{\cdot\cdot},\quad i=1,2 \tag{1.1}$$

with the boundary conditions on the surface $x_3=0$

$$\mathbf{N}\cdot\boldsymbol{\theta}^{(1)}=\mathbf{q}^{(1)},\quad x_1,x_2\in\Omega \tag{1.2}$$

and the displacement and stress continuity conditions on the interfacial surface of the medium $x_3=h$.

The quantities with superscript 1 in (1.1), (1.2) and below refer to the layer, and those with superscript 2 refer to the half-space $\mathbf{u}^{(i)}=\{u_1^{(i)},u_2^{(i)},u_3^{(i)}\}$, $\mathbf{q}^{(i)}=\{q_1^{(i)},q_2^{(i)},q_3^{(i)}\}$ are the displacement and stress vectors of the layer $(i=1)$ of the half-space $(i=2)$ respectively, $\rho^{(i)}$ is the material density, and $\mathbf{N}$ is the vector normal to the surface. In the case under consideration $\mathbf{N}=\{0,0,1\}$ and $\boldsymbol{\theta}^{(i)}$ is a tensor of fourth rank that is represented in the form of the sum of a symmetric tensor $\mathbf{P}^{(i)}$ and an antisymmetric tensor $\mathbf{U}^{(i)}$ /9/.

The tensor $\mathbf{U}^{(i)}$ is independent of the material properties and is represented in terms of the initial stress tensor $\mathbf{T}^{(i)}$ and the symmetric strain tensor $\boldsymbol{\varepsilon}^{(i)}$ and skew-symmetric strain tensor $\boldsymbol{\Omega}^{(i)}$ /9/

$$\mathbf{U}^{(i)}=\frac{1}{2}(\mathbf{T}^{(i)}\cdot\boldsymbol{\varepsilon}^{(i)}-\boldsymbol{\varepsilon}^{(i)}\cdot\mathbf{T}^{(i)})-\mathbf{T}^{(i)}\cdot\boldsymbol{\Omega}^{(i)} \tag{1.3}$$

The tensor $P^{(i)}$ is independent of the initial stresses and in the case of small initial strains has the form ($\lambda^{(i)}$, $\mu^{(i)}$ and Lame parameters, and E is the unit tensor)

$$\mathbf{P}^{(i)}(\boldsymbol{\varepsilon}^{(i)})=\lambda^{(i)}(x_3)\,\mathrm{tr}\,\boldsymbol{\varepsilon}^{(i)}\cdot\mathbf{E}+2\mu^{(i)}(x_3)\,\boldsymbol{\varepsilon}^{(i)} \tag{1.4}$$

We shall later assume that the initial state of stress is determined by the initial stress tensor $\mathbf{T}^{(i)}$ with the components

$$T_{ks}^{(i)}=\sigma_{ks}^{0(i)},\quad k,s=1,2,3 \tag{1.5}$$

Here δ_{ks} is the Kronecker delta, where $\sigma_{kk}^{0(2)}=\mathrm{const}\ (k=1,2,3)$, and $\sigma_{kk}^{0(1)}=\sigma_{kk}^{0(1)}(x_3)$ are arbitrary fairly smooth functions of the coordinate x_3.

Taking (1.3)-(1.5) into account we represent the boundary value Problem (1.1) and (1.2) in the following way

$$\partial\theta_{ks}^{(i)}/\partial x_k = \rho^{(i)}\ddot{u}_s^{(i)} \tag{1.6}$$
$$N_s\theta_{ks}^{(1)} = q_k^{(1)}, \quad x_3 = 0; \quad u_k^{(1)} = u_k^{(2)}, \quad \theta_{k3}^{(1)} = \theta_{k3}^{(2)}, \quad x_3 = h$$
$$k, s = 1, 2, 3; \ i = 1, 2$$

Here

$$\theta_{ks}^{(i)} = a_{ks}^{(i)} \frac{\partial u_k^{(i)}}{\partial x_s} + b_{ks}^{(i)} \frac{\partial u_s^{(i)}}{\partial x_k}, \quad k \neq s$$
$$\theta_{kk} = (\lambda^{(i)} + 2\mu^{(i)}) \frac{\partial u_k^{(i)}}{\partial x_k} + \lambda^{(i)} \left(\frac{\partial u_s^{(i)}}{\partial x_s} + \frac{\partial u_p^{(i)}}{\partial x_p} \right)$$
$$k \neq p \neq s; \quad k, s, p = 1, 2, 3 \tag{1.7}$$
$$a_{ks}^{(i)} = \mu^{(i)} - {}^1/_4\sigma_{kk}^{0(i)} - {}^1/_4\sigma_{ss}^{0(i)}, \quad b_{ks}^{(i)} = \mu^{(i)} + {}^3/_4\sigma_{kk}^{0(i)} - {}^1/_4\sigma_{ss}^{0(i)}$$

Let us recall that the coefficients $a_{ks}^{(i)}$, $b_{ks}^{(i)}$, $\lambda^{(i)}$, $\mu^{(i)}$, $\sigma_{kk}^{0(i)}$ in (1.6) and (1.7) constant for $i = 2$ and are arbitrary fairly smooth functions of the coordinate x_3 for $i = 1$.

2. Applying a two-dimensional Fourier transformation in the coordinates x_1, x_2 (α, β are transformation parameters) to the boundary value Problem (1.6), it can be reduced to the form /7/

$$\mathbf{Y}^{(i)\prime} = \mathbf{M}^{(i)}(\alpha, \beta, x_3)\,\mathbf{Y}^{(i)}, \quad i = 1, 2 \tag{2.1}$$

$$\mathbf{B}^{(1)}(\alpha, \beta, x_3)\cdot\mathbf{Y}^{(1)} = \mathrm{S}^{(1)}, \quad x_3 = 0$$
$$\mathbf{B}^{(0)}(\alpha, \beta, x_3)\cdot\mathbf{Y}^{(0)} = 0, \quad x_3 = h \tag{2.2}$$

Here

$$\mathbf{Y}^{(k)} = \uparrow \{i\alpha V_1^{(k)}, i\beta V_2^{(k)}, V_3^{(k)}, i\alpha U_1^{(k)}, i\beta U_2^{(k)}, U_3^{(k)}\}, \quad k = 1, 2$$
$$\mathbf{Y}^{(0)} = \uparrow \{i\alpha V_1^{(1)}, i\beta V_2^{(1)}, V_3^{(1)}, i\alpha U_1^{(1)}, i\beta U_2^{(1)}, U_3^{(1)}, i\alpha V_1^{(2)}, i\beta V_2^{(2)}, V_3^{(2)}, i\alpha U_1^{(2)}, i\beta U_2^{(2)}, U_3^{(2)}\}$$
$$\mathrm{S}^{(1)} = \uparrow \{i\alpha Q_1, i\beta Q_2, Q_3\}, \quad V_k^{(i)} = dU_k^{(i)}/dx_3$$

where $U_k^{(i)}$, Q_k are transforms of the Fourier components of the displacement and stress vectors, respectively. The matrices $\mathbf{M}^{(i)}(\alpha, \beta, x_3)$ $(i = 1, 2)$ have the dimensions 6x6 and $\mathbf{B}^{(0)}(\alpha, \beta, x_3)$ and $\mathbf{B}^{(1)}(\alpha, \beta, x_3)$, have the dimensions 12x6 and 6x3, respectively. The elements of the matrices $\mathbf{M}^{(1)}(\alpha, \beta, x_3)$, $\mathbf{B}^{(1)}(\alpha, \beta, x_3)$ and $\mathbf{B}^{(0)}(\alpha, \beta, x_3)$ are functions of the coordinate x_3 that are determined by the nature of the change in the elastic parameters and the initial stresses in the layer. The components of the matrix $\mathbf{M}^{(2)}$ are independent of x_3 and are determined by the elastic properties of the homogeneous half-space, and the form and intensity of the initial state of stress.

The solution of system (2.1) for the half-space $(i = 2)$ can be represented in the form

$$Y_k^{(2)}(\alpha, \beta, x_3) = \sum_{i=1}^{3} c_{i+6}(\alpha, \beta)\, m_{ik}(\alpha, \beta)\, e^{-\sigma_i(\alpha, \beta)x_3}, \quad k = 1, 2, 3 \tag{2.3}$$

Here σ_i $(\mathrm{Re}\,\sigma_i \geqslant 0,\ \mathrm{Im}\,\sigma_i \leqslant 0,\ i = 1, 2, 3)$ are solutions of the characteristic equation

$$|\mathbf{M}^{(2)}(\alpha, \beta) - \sigma\mathbf{E}| = 0 \tag{2.4}$$

($\mathbf{E}$ is the unit matrix) and $m_{ik}(\alpha, \beta)$ are coupling coefficients determined from the characteristic equation.

The six linearly independent solutions of system (2.1) can be obtained numerically by the Runge-Kutta, Adams, etc. methods for fixed values of the parameters α, β. We assume these solutions with the initial conditions $y_{ik}^{(1)}(\alpha, \beta, 0) = \delta_{ik}$ to be constructed and have the form

$$Y_k^{(1)} = \sum_{i=1}^{6} c_i(\alpha, \beta)\, y_{ik}^{(1)}(\alpha, \beta, x_3), \quad k = 1, 2, \ldots, 6 \tag{2.5}$$

The unknowns $c_i(\alpha, \beta)$ $(i = 1, 2, ..9)$ taking part in the representations (2.3) and (2.5) are found when the boundary conditions (2.2) satisfy (2.3) and (2.5). After using the limit absorption principle /1, 7/ and the inverse Fourier transformation, we obtain

$$\mathbf{u}^{(i)}(x_1, x_2, x_3) = \frac{1}{2\pi}\iint_{\Omega} \mathbf{k}^{(i)}(x_1-\xi, x_2-\eta, x_3)\,\mathbf{q}^{(i)}(\xi,\eta)\,d\xi\,d\eta$$

$$\mathbf{k}^{(i)}(s, t, x_3) = \int_{\Gamma_1}\int_{\Gamma_2} \mathbf{K}^{(i)}(\alpha, \beta, x_3)\,e^{-i(\alpha s+\beta t)}\,d\alpha\,d\beta \tag{2.6}$$

The contours Γ_1 and Γ_2 are selected in conformity with the limit absorption principle /1, 2/ and circumvent the singularities of the matrix function $K(\alpha, \beta, x_3)$ in a special manner. The right-hand side of the first equality in (2.6) determines the displacement vector of an arbitrary point of the layer $0 \leqslant x_3 \leqslant h$, $|x_1|$, $|x_2| < \infty$ $(i = 1)$ or half-space $x_3 \geqslant h$, $|x_1|$, $|x_2| < \infty$ $(i = 2)$.

3. Assuming $x_3 = 0$ in (2.6) and considering the stamp displacement $u^{(1)}(x_1, x_2, 0) = u^{\circ}(x_1, x_2)$ to be known, we arrive at a system of integral equations in the unknown contact stress distribution vector-function

$$u^{\circ}(x_1, x_2) = \frac{1}{2\pi}\iint_{\Omega} \mathbf{k}^{\circ}(x_1-\xi, x_2-\eta)\,\mathbf{q}(\xi,\eta)\,d\xi, d\eta, \quad x_1, x_2 \in \Omega \tag{3.1}$$

$$\mathbf{k}^{\circ}(s, t) = \mathbf{k}(s, t, 0) = \int_{\Gamma_1}\int_{\Gamma_2} \mathbf{K}^{\circ}(\alpha, \beta)\,e^{-i(\alpha s+\beta t)}\,d\alpha\,d\beta \tag{3.2}$$

The matrix function $\mathbf{K}^{\circ}(\alpha, \beta) = \mathbf{K}^{(1)}(\alpha, \beta, 0)$ has a representation characteristic for dynamic contact problems of elasticity theory /1, 2/

$$\mathbf{K}^{\circ}(\alpha, \beta) = \begin{vmatrix} \alpha^2 M + \beta^2 N & i\beta(M-N) & i\alpha S \\ \alpha\beta(M-N) & \alpha^2 N + \beta^2 M & i\beta T \\ i\alpha S & i\beta T & R \end{vmatrix} \tag{3.3}$$

with elements M, N, S, T, R, dependent on the parameters $u = \sqrt{\alpha^2 + \beta^2}$, of frequency ω, and the kind of dependence of the elastic parameters and the initial state of stress of the layer on the coordinates. The functions M, N, S, T and R can be represented in analytic form only in special cases of the dependences of the parameters $\lambda^{(1)}$, $\mu^{(1)}$, $\rho^{(1)}$, as well as $\overset{\circ}{\sigma}{}_{kk}^{(1)}$ on x_3 (for instance, $\lambda^{(1)} = \lambda_0^{(1)}e^{\gamma x_3}$, $\mu^{(1)} = \mu_0^{(1)}e^{\gamma x_3}$, $\rho^{(1)} = \rho_0^{(1)}e^{\gamma x_3}$, $\overset{\circ}{\sigma}{}_{kk}^{(1)} = \sigma_{kk}e^{\gamma x_3}$ or analogous) allowing of an analytic solution of system (2.1) for $i = 1$.

In the general case the functions M, N, S, T and R can only be obtained numerically, which makes an investigation of their properties difficult, a knowledge of which is needed for the subsequent analysis and construction of the solution of the integral equation. Certain properties of these functions (evenness and meromorphicity) and established from the form of the analytic dependences of the coefficients of system (2.1) on the parameters α and β. By virtue of the homogeneity of the half-space, the branch points are determined fairly easily and, therefore, the domain of analyticity of the function as well. The investigation of their asymptotic behaviour as $\alpha, \beta \to \infty$ is important.

Following /7/, we reduce system (2.1) to a system of differential equations with a small parameter in the highest derivative. An investigation of this system enables us to write a representation of the function-elements of the matrix $\mathbf{K}^{\circ}$ for $u = \sqrt{\alpha^2 + \beta^2}$ in the form $(m, n, s, t, r$ are constant coefficients)

$$M = mu^{-3} + O(u^{-5}), \quad N = nu^{-3} + O(u^{-5})$$
$$R = ru^{-1} + O(u^{-3}), \quad S = su^{-1} + O(u^{-3}), \quad T = tu^{-2} + O(u^{-4}) \tag{3.4}$$

Further study of the properties of the elements of the matrix function $\mathbf{K}^{\circ}$ is possible only when the specific parameters of the inhomogeneity of the medium are given and utilizing numerical methods. In particular, a detailed analysis of the distribution of the zeros and poles of the elements of $\mathbf{K}^{\circ}$, taking (3.4) into account enables us to approximate the matrix function $\mathbf{K}^{\circ}$ by a polynomial function with all the most important properties conserved. Furthermore, by using the method of approximate factorization of the functions and matrix-functions /1, 2/ a solution can be constructed for the system of integral Eqs.(3.1) and the influence of the system parameters of an inhomogeneous layer-homogeneous prestressed half-space can be investigated in a wave field on the surface of the medium.

Certain characteristics of the wave field, particularly the phase characteristics of the surface wave velocity can be investigated by analysing the properties of the kernel of the integral operator of system (3.1).

Without loss of generality, we set

$$\mu^{(1)} = \mu_0 e^{\theta x_3}, \quad \lambda^{(1)} = \lambda_0 e^{\gamma x_3}, \quad \rho^{(1)} = \rho_0 e^{\delta x_3}$$
$$\mu^{(2)} = k_1 \mu_0, \quad \lambda^{(2)} = k_2 \lambda_0, \quad \rho^{(2)} = k_3 \rho_0 \tag{3.5}$$

We furthermore assume that the initial stresses in the layer are due to the effect of gravity

$$\sigma_{33}^{0(1)} = k_0 e^{\delta x_3}, \quad k_0 = -\rho_0 g/\delta, \quad \delta \neq 0 \tag{3.6}$$

(g is the acceleration due to gravity). The constant coefficients θ, γ, δ, k_1, k_2, k_3 in (3.5) and (3.6), enable the properties of the medium to be varied over a wide range.

Numerical analysis showed that for $k_1 = k_2 = K > 1$ over a fairly wide range of variation of the parameters θ, γ, δ the elements of the matrix-function K° have a denumerable set of zeros and poles when they alternate strictly for the elements $M(u)$ and $R(u)$. For fixed ω, the zeros and poles lie between $V_s^{(1)}$ and $V_s^{(2)}$ (the transverse velocities in the layer and the half-space, respectively).

Graphs of the phase velocities of the surface elastic waves $V_i(\varkappa_2)$ (i is the number of the surface wave mode, $\varkappa_2$ is the generalized frequency parameter $\varkappa_2 = \omega h (\rho_0/\mu_0)^{1/2}$ are presented in Figs.1 and 2 for different K (curves 1-5 correspond to $K = 1, 10, 50, 100, 500$) and $\theta = \gamma = 0$, $\sigma_{jj}^{\circ(1)} = 0$ $(j = 1, 3)$. As K increases the phase velocity of each mode increases while the number of modes at a fixed frequency decreased. Each mode of vibration exists for $\varkappa_2 > \varkappa_{2i}$. The frequency $\varkappa_{2i}$ is called the boundary frequency of the i-th mode /10/.

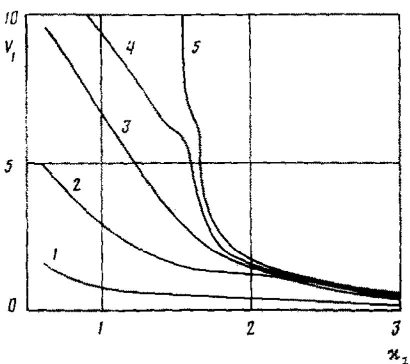

Fig.1

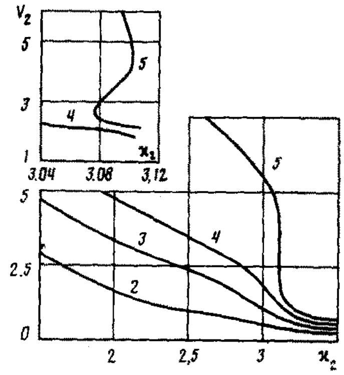

Fig.2

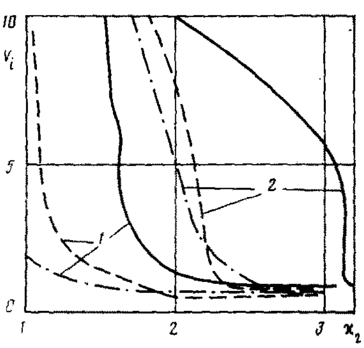

Fig.3

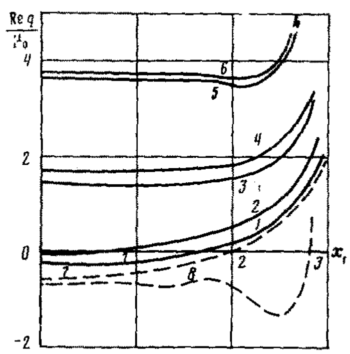

Fig.4

As $\varkappa_2 \to \infty$ the phase velocities decrease, their value $V_i \to V_s^{(1)}$ for all K. For $K \gg 1$ a range of variation of $\varkappa_2$ (Fig.2) is observed in which the phase velocities increase as the frequency increases, poles occur that do not alternate with zeros associated with the appearance of a backward wave. The dispersion properties of the problem become analogous to the problem of stamp vibrations on a layer which adheres rigidly to a non-deformable base.

A change in the parameters of the layer inhomogeneity θ, δ, γ influence the surface field differently. Graphs of $V_i(\varkappa_2)$ are presented in Fig.3 for $\delta = 0, 2, 3$ $(\gamma = \theta = 0,\ K = 500)$ (the solid, dashed, and dash-dot lines, respectively; the numbers 1 and 2 denote the first and second modes of vibration). As δ increases the boundary frequency $\varkappa_{2i}$ of the i-th mode decreases. As the frequency (the parameter $\varkappa_2$) increases, the phase velocities decrease more rapidly the greater the parameter δ, and the number of waves themselves (modes) at a fixed frequency increases.

4. To illustrate the method described above for investigating the influence of the parameters of an inhomogeneous medium on a wave field, we will consider the problem of the vibrations of a rigid stamp occupying a strip $|x_1| \leqslant a, |x_2| < \infty$ in planform as an example. We assume there is no friction in the contact domain (to shorten the calculations), i.e.

$$\mathbf{q}^{(1)}(x_1, x_2, 0) = \begin{cases} \{0, 0, q^{(1)}(x_1)\}, & |x_1| \leqslant a, |x_2| < \infty \\ 0, & |x_1| > a, |x_2| < \infty \end{cases} \tag{4.1}$$

Within the framework of these assumptions we must consider the plane boundary value Problem (1.6) about wave field excitation in an inhomogeneous medium with the difference that the subscripts k, s in (1.6) and (1.7) take the values 1 and 3. Using the Fourier transform in x_1 (α is the transformation parameter) we can reduce Problem (1.6) to the form (2.1) and (2.2) without special difficulties, with the vectors

$$\mathbf{Y}^{(i)} = \uparrow \{i\alpha V_1^{(i)}, V_3^{(i)}, i\alpha U_1^{(i)}, U_3^{(i)}\}, \quad i = 1, 2$$

$$\mathbf{S}^{(1)} = \uparrow \{0, 0, Q^{(1)}\}$$

$$\mathbf{Y}^{(0)} = \uparrow \{i\alpha V_1^{(1)}, V_3^{(1)}, i\alpha U_1^{(1)}, U_3^{(1)}, i\alpha V_1^{(2)}, V_3^{(2)}, i\alpha U_1^{(2)}, U_3^{(2)}\}$$

and the matrices $\mathbf{M}^{(i)}(\alpha, x_3)$ $(i = 1, 2)$ of dimensions 4x4, and $\mathbf{B}^c(\alpha, x_3)$ and $\mathbf{B}^{(1)}(\alpha, x_3)$ of dimensions 8x4 and 4x2, respectively.

We will represent the solution for the half-space (by analogy with (2.3)) in the form

$$Y_k^{(2)}(\alpha, x_3) = \sum_{j=1}^{2} c_{j+4}(\alpha)\, m_{jk}(\alpha)\, e^{-\sigma_j(\alpha) x_3}, \quad k = 1, 3$$

$$\sigma_{1,2} = [(-D_2 \pm \sqrt{D_2^2 - 4D_1D_3})/(2D_1)]^{1/2} \tag{4.2}$$

$$D_1 = A_1A_6, \quad D_2 = S_1A_1 + S_2A_6 + \alpha^2A_5^2, \quad D_3 = S_1S_2$$

$$A_1 = b_{31}^{(2)}, \quad A_2 = b_{13}^{(2)}, \quad A_6 = \lambda^{(2)} + 2\mu^{(2)}, \quad A_5 = \lambda^{(2)} + a_{13}^{(2)} \tag{4.3}$$

$$S_1 = \rho^{(2)}\omega^2 - A_2\alpha^2, \quad S_2 = \rho^{(2)}\omega^2 - A_6\alpha^2$$

$$m_{j3} = -\sigma_j[D_1\sigma_j^2 + A_6S_2 + \alpha^2A_5^2]/(S_1A_5), \quad m_{j1} = 1, \quad j = 1, 2 \tag{4.4}$$

Assuming linearly independent solutions for the layer $(i = 1)$ constructed (there are four in the plane case and they can be obtained numerically for the initial conditions $y_{kj}^{(1)}(\alpha, 0) = \delta_{jk}$) and having the form

$$Y_k^{(1)}(\alpha, x_3) = \sum_{j=1}^{4} c_j(\alpha)\, y_{kj}^{(1)}(\alpha, x_3), \quad k = 1, 2, 3, 4 \tag{4.5}$$

we arrive at a system of six equations to determine $c_k(\alpha)$ $(k = 1, 2, \ldots, 6)$

$$\sum_{j=1}^{6} A_{kj}c_j(\alpha) = T_k(\alpha) \tag{4.6}$$

$$T_1 = T_2 = T_4 = T_5 = T_6 = 0, \quad T_3 = Q_3(\alpha)$$

$$A_{11} = b_{31}^{(1)}(0), \quad A_{12} = A_{13} = A_{15} = A_{16} = A_{21} = A_{24} = A_{25} = A_{26} = 0$$

$$A_{14} = a_{31}^{(1)}(0), \quad A_{mi} = y_{mi}^{(1)}(\alpha, h), \quad m = 3, 4; \quad i = 1, 2, 3, 4$$

$$A_{5i} = b_{31}^{(1)}(h)\, y_{1i}^{(1)}(\alpha, h) + a_{31}^{(1)}(h)\, y_{4i}^{(1)}(\alpha, h) \tag{4.7}$$

$$A_{6i} = (\lambda^{(1)}(h) + 2\mu^{(1)}(h))\, y_{2i}^{(1)}(\alpha, h) - \alpha^2\lambda^{(1)}(h)\, y_{4i}^{(1)}(\alpha, h), \quad i = 1, 2, 3, 4$$

$$A_{35} = -e^{-\sigma_1 h}, \quad A_{36} = -e^{-\sigma_2 h}, \quad A_{45} = -m_{12}e^{-\sigma_1 h}$$

$$A_{46} = -m_{22}e^{-\sigma_2 h}, \quad A_{55} = -(a_{31}^{(2)}m_{12} - b_{31}^{(2)}\sigma_1)\, e^{-\sigma_1 h}$$

$$A_{56} = -(a_{31}^{(2)}m_{22} - b_{31}^{(2)}\sigma_2)\, e^{-\sigma_2 h}$$

$$A_{65} = -[(\lambda^{(2)} + 2\mu^{(2)})\, m_{12}\sigma_1 + \alpha^2\lambda^{(2)}]\, e^{-\sigma_1 h}, \quad A_{66} = -[(\lambda^{(2)} + 2\mu^{(2)})\, m_{22}\sigma_2 + \alpha^2\lambda^{(2)}]\, e^{-\sigma_2 h}$$

It is here taken into account that the components $V_1{}^2$ and $V_3{}^2$ are expressed in a linear manner in terms of the components $U_1^{(2)}$ and $U_3^{(2)}$ by virtue of (4.2).

The solution of system (4.6) has the form

$$c_i(\alpha) = Q_3(\alpha)B_{i3}/\Delta(\alpha) \tag{4.8}$$

where B_{ij} are the elements of the matrix consisting of the cofactors of the elements A_{ij} of the matrix A.

Taking (4.2), (4.5) and (4.8) into account, we can represent the displacement vector of an arbitrary point of the medium in the form

$$u_3^{(i)}(x_1, x_3) = \frac{1}{2\pi}\int_{-a}^{a} k^{(i)}(x_1-\xi, x_3)\, q(\xi)\, d\xi \tag{4.9}$$

$$k^{(i)}(s, x_3) = \int_{\Gamma} R^{(i)}(\alpha, x_3)\, e^{-i\alpha s}\, d\alpha$$

In the case of a contact problem it is necessary to set $i = 1, x_3 = 0$ in (4.9). Assuming $u_3^{(1)}(x_1, 0)$ to be a known function we obtain an integral equation in the unknown contact stress distribution functions $q^{(1)}(x_1)$ with the kernel $R^{(1)}(\alpha, 0) = R_0(\alpha)$.

The solution of this integral equation is constructed by the approximate factorization method. To this end we approximate $R_0(\alpha)$ by the function

$$R^*(\alpha) = r^{(1)}(\alpha^2 + B^2)^{-1/2} \prod_{k=1}^{N} (\alpha^2 - z_k^2)(\alpha^2 - \zeta_k^2)^{-1} \tag{4.10}$$

Here $r^{(1)}$ is a constant occurring in (3.4), z_k $(k = 1, 2, \ldots, n_1)$, ζ_k $(k = 1, 2, \ldots, m_1)$ are real zeros and poles $R(\alpha)$; B, z_k $(k = n_1 + 1, \ldots, N)$, ζ_k $(k = m_1 + 1, \ldots, N)$ are approximation parameters that are determined from the condition for best approximation of $R_0(\alpha)$ by the functions $R^*(\alpha)$ on the real axis.

The representation of the functions $q^{(1)}(x_1), |x_1| \leqslant a, u^{(1)}(x_1)$ when using an approximation of the form (4.10) is described in the literature (/1, 2, 5, 7, 11/, say) and is not presented here.

Graphs of the function $\operatorname{Re} q(x_1)/\mu_0$ are shown in Fig.4 for different values of $k_1 = k_2 = K$ and δ for $\theta = \gamma = 0$ in (3.5). The solid lines represent $\operatorname{Re} q(x_1)/\mu_0$ for the case $\sigma_{jj}^{0(I)} = 0\ (j = 1, 3)$ (a homogeneous layer) and $K = 1, 2, 7, 10, 100, 500$ (denoted by the numbers *1, 2, ..., 6*). It is seen that as K increases the amplitude to the contact stresses increases, where it is transformed from parabolic $(K = 1)$ to a form characteristic for static problems for $K \gg 1$. This can be explained by the fact that there are no real zeros and poles at the given frequency $\varkappa_2 = 0.8$ for $K \gg 1$, i.e., the problem becomes quasistatic. Contact stresses for different values of δ are represented by the dashed lines.

The numbers *7* and *8* denote graphs of $\operatorname{Re} q(x_1)/\mu_0$ for $\delta = 1, 2, K = 1$, and $\gamma = \theta = 0$. As δ increases the amplitude of the stress drops, and the diagram is transformed from parabolic into saddle-shaped, and the stress oscillations are magnified under the stamp.

REFERENCES

1. VOROVICH I.I. and BABESHKO V.A., Dynamic Mixed Problems of Elasticity Theory for Non-classical Domains. Nauka, Moscow, 1979.
2. BABESHKO V.A., The Generalized Factorization Method and Spatial Dynamic Mixed Problems of Elasticity Theory. Nauka, Moscow, 1984.
3. GUZ A.N., Elastic Waves in Bodies with Initial Stresses, Naukova Dumka, Kiev, 1986.
4. GREEN A.E., RIVLIN R.S. and SHIELD R.T., General theory of small elastic deformations superposed on finite elastic deformations, Proc. Roy. Soc., Ser. A, 211, 1104, 1952.
5. BABESHKO V.A., GLUSHKOV E.V. and GLUSHKOVA N.V., Methods of constructing Green's matrix of a stratified elastic half-space, Zh. vychisl. Mat. mat. Fiz., 27, 1, 1987.
6. GLUSHKOV E.V., Energy flux for harmonic oscillations of multilayered elastic media, Proceedings of the Sixth All-Union Congress on Theoretical and Applied Mechanics, Tashkent, 1986.
7. ANAN'YEV I.V., KALINCHUK V.V. and POLYAKOVA I.B., On wave excitation by a vibrating stamp in a medium with inhomogeneous initial stresses, PMM, 47, 3, 1983.
8. LUR'YE A.I., Theory of Elasticity, Nauka, Moscow, 1970.
9. ZUBOV L.M., Theory of small deformations of prestressed thin shells, PMM, 40, 1, 1976.
10. AKI K. and RICHARDS P., Quantitative Seismology, 1, Mir, Moscow, 1983.
11. KALINCHUK V.V. and POLYAKOVA I.B., On the excitation of a prestressed cylinder, PMM, 45, 2, 1981.

Translated by M.D.F.